CHAPTER 4: Integration

Duration: 3 weeks or 13,5 hours

BACKGROUND

This topic has links with the chapter on differentiation.

TOPIC OVERVIEW

Learners are expected to:

- Understand the concept of integration as both a definite integral and an indefinite integral
- Apply standard form of integrals as a converse of differentiation
- Integrate functions of the forms $f(x) = kx^n$ with $n \in R$ and $n \neq -1$; $f(x) = \frac{k}{x}$ and $f(x) = kan^x$ with $a \geq 0$ and $k, a \in R$
- Apply integration to determine the magnitude of an area included by a curve and the x-axis or by a curve, the x-axis and the ordinates $x = a$ and $x = b$ where $a, b \in Z$

PRIOR KNOWLEDGE

Learners must have knowledge of:

- Functions
- Differentiation
- Area
- Indices
- Limits

SOLUTIONS

Exercise 1

1.1 $\int 5\ dx = \int 5x^0\ dx$

$$= 5\left(\frac{x^{0+1}}{0+1}\right) + C$$

$$= 5\left(\frac{x}{1}\right) + C$$

$$= 5x + C$$

1.2 $\int -3\ dx = -3x + C$

1.3 $\int x^3 dx = \frac{x^{3+1}}{3+1} + C$

1.4 $\int 6x^5\,dx = 6\left(\frac{x^{5+1}}{5+1}\right) + C$

$$= \frac{x^4}{4} + C$$

1.5 $\int 8x^7 dx = x^8 + C$

1.6 $\int -x\,dx = -1\int x\,dx$

$$= -1\left(\frac{x^{1+1}}{1+1}\right) + C$$

$$= -1\left(\frac{x^2}{2}\right) + C$$

$$= -\frac{x^2}{2} + C$$

1.7 $\int x^4\,dx = \frac{x^5}{5} + C$

1.8 $\int -3x\,dx = -\frac{3}{2}x^2 + C$

1.9 $\int \frac{1}{2}x\,dx = \frac{1}{2}\left(\frac{x^{1+1}}{1+1}\right) + C$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

$$= \frac{x^2}{4} + C$$

$$= 6\left(\frac{x^6}{6}\right) + C$$

$$= x^6 + C$$

Exercise 2

2.1 $\int (x+1)\,dx = \int x\ dx + \int 1\ dx$

$$= \frac{x^2}{2} + x + C$$

2.2 $\int (-x+1)\,dx = \frac{-x^2}{2} + x + C$

2.3 $\int (3x+2)\,dx = \int 3x\ dx + \int 2dx$

$$= \frac{3x^2}{2} + 2x + C$$

2.4 $\int \left(1-x^2\right) dx = x - \frac{x^3}{3} + C$

2.5 $\int \left(1-x^3\right) dx = \int 1dx - \int x^3 dx$

$$= x - \frac{x^4}{4} + C$$

2.6 $\int \left(x^2 - 2x + 1\right) dx = \int x^2\ dx - \int 2x\ dx + \int 1dx$

$$= \frac{x^3}{3} - x^2 + x + C$$

2.7 $\int \left(x^3 - 2x^2 + x - 5\right) dx = \int x^3\ dx - \int 2x^2\ dx + \int x\ dx - \int 5dx$

$$= \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} - 5x + C$$

2.8 $\int \left(3x^4 - x^5\right) dx = \int 3x^4\ dx - \int x^5\ dx$

$$= \frac{3x^5}{5} - \frac{x^6}{6} + C$$

2.9 $\int \left(5 + 3x + x^2\right) dx = \int 5dx + \int 3x\ dx + \int x^2\ dx$

$$= 5x + \frac{3x^2}{2} + \frac{x^3}{3} + C$$

2.10 $\int \left(-3x^2 + \frac{1}{2}x + 1\right) dx = -x^3 + \frac{x^2}{4} + x + C$

Exercise 3

3.1 $\int \frac{7}{x} dx = 7\int \frac{1}{x}$

$= \ln|x| + C$

3.2 $\int \frac{100}{x} = 100\int \frac{1}{x}$

$= 100\ln|x| + C$

3.3 $\int \frac{23}{x} = 23\ln|x| + C$

3.4 $\int \frac{\sqrt{3}}{x} = \sqrt{3}\ln|x| + C$

3.5 $\int \frac{\sqrt{5}}{x} = \sqrt{5}\ln|x| + C$

3.6 $\int \frac{\sqrt{11}}{x} = \sqrt{11}\ln|x| + C$

3.7 $\int \frac{2}{3x} dx = \int \frac{2}{3}\left(\frac{1}{x}\right) dx$

$= \frac{2}{3}\int \frac{1}{x} dx$

$= \frac{2}{3}\ln|x| + C$

3.8 $\int \frac{2}{5x} dx = \frac{2}{5}\ln|x| + C$

3.9 $\int \frac{-1}{2x} dx = -\frac{1}{2}\ln|x| + C$

Exercise 4

4.1 $\int_0^2 (x^2 + x)\, dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$

$= \frac{2^3}{3} + \frac{2^2}{2}$

$= \frac{14}{3}$

4.2 $\int_1^3 (3u^3 - 1)\, du$

$= \left[\frac{3u^4}{4} - u\right]_1^3$

$= \left(\frac{3(3)^4}{4} - 3\right) - \left(\frac{3(1)^4}{4} - 1\right)$

$= \frac{231}{4} - \left(-\frac{1}{4}\right)$

$= 58$

4.3 $\int_{-2}^{-1} (x^3 - 2x)\, dx$

$= \left[\frac{x^4}{4} - x^2\right]_{-2}^{-1}$

$= \left(\frac{(-1)^4}{4} - (-1)^2\right) - (\frac{(-2)^4}{4} - (-2)^2)$

$= -\frac{3}{4} - 0 = -\frac{3}{4}$

4.4 $\int_{-2}^{1} (6x^2 - 5x + 2)\, dx$

$= \left[\frac{6x^3}{3} - \frac{5x^2}{2} + 2x\right]_{-2}^{1}$

$= \left[2x^3 - \frac{5x^2}{2} + 2x\right]_{-2}^{1}$

$= \left(2(1^3) - \frac{5(1)^2}{2} + 2(1)\right) - \left(2(-2^3) - \frac{5(-2)^2}{2} + 2(-2)\right)$

$= \frac{3}{2} - (-30)$

$= 31\frac{1}{2}$

4.5 $\int_0^4 \sqrt{t}\,(t+2)dt$

$$= \int_0^4 t^{\frac{1}{2}}\,(t+2)dt$$

$$= \int_0^4 (t^{\frac{3}{2}} + 2t^{\frac{1}{2}})dt$$

$$= \left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{2t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$= \left[\frac{2t^{\frac{5}{2}}}{5} + \frac{4t^{\frac{3}{2}}}{3}\right]_0^4$$

$$= \frac{2(4)^{\frac{5}{2}}}{5} + \frac{4(4)^{\frac{3}{2}}}{3} = \frac{352}{15}$$

4.6 $\int_0^1 \left(4x - 6x^{\frac{2}{3}}\right) dx$

$$= \left[\frac{4x^2}{2} - \frac{6x^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^1$$

$$= \left[2x^2 - \frac{18x^{\frac{5}{3}}}{5}\right]_0^1$$

$$= 2(1) - \frac{18(1)^{\frac{5}{3}}}{5} = -\frac{8}{5}$$

4.7 $\int_{-2}^0 \left(\frac{1}{3}x^2 - 2\right) dx$

$$= \left[\left(\frac{1}{3}\right)\frac{x^3}{3} - 2x\right]_{-2}^0$$

$$= \left[\frac{x^3}{9} - 2x\right]_{-2}^0$$

$$= 0 - \left(\frac{-2^3}{9} - 2(-2)\right)$$

$$= \frac{-10}{9}$$

4.8 $\int_2^3 \left(\frac{1}{2}x^4 + x\right) dx$

$= \left[\frac{x^5}{10} - \frac{x^2}{2}\right]_2^3$

$= \left(\frac{3^5}{10} + \frac{3^2}{2}\right) - \left(\frac{2^5}{10} + \frac{2^2}{2}\right)$

$= \frac{144}{5} - \frac{26}{5}$

$= \frac{118}{5}$

Exercise 5

5.1 $\int_{-2}^{2} (-4 + x^2)\, dx$

$= \left[-4x + \frac{x^3}{3}\right]_{-2}^{2}$

$= \left[-4x + \frac{x^3}{3}\right]_{-2}^{0} + \left[-4x + \frac{x^3}{3}\right]_{0}^{2}$

$= \left|-4(-2) + \frac{-2^3}{3}\right| + \left|-4(2) + \frac{2^2}{3}\right|$

$= \left|\frac{16}{3}\right| + \left|\frac{-16}{3}\right|$

$= \frac{32}{3}$ square units

5.2 $\int_0^2 (-x^2 + 4)\, d$

$= \left[-\frac{x^3}{3} + 4x\right]_0^2$

$= \left|-\frac{2^3}{3} + 4(2)\right| + |0|$

$= -\frac{8}{3} + 8$

$= \frac{16}{3}$ square units

5.3 $\int_{-3}^{2}(2x+1)\,dx$

$= [x^2 + x]_{-3}^{2}$

$= [x^2 + x]_{-3}^{\frac{-1}{2}} \quad + \quad [x^2 + x]_{\frac{-1}{2}}^{2}$

$= \left|\left((-\frac{1}{2})^2 - \frac{1}{2}\right) - ((-3)^2 - 3)\right| + |(2^2) + 2) - (|\left((-\frac{1}{2})^2 - \frac{1}{2}\right)$

$= \left|\frac{-25}{4}\right| + \left|\frac{25}{4}\right|$

$= \frac{50}{4}$

$= \frac{25}{2}$ unit squared

5.4 $\int_{2}^{3}(x^2 - 6x + 5)dx$

$= \left[\frac{x^3}{3} + 3x^2 + 5x\right]_1^5$

$= \left|\left(\left(\frac{5^3}{3}\right) - 3(5^2) + 5(5)\right) - (\frac{1}{3} - 3 + 5)\right|$

$= \left|\frac{125}{3} - 75 + 25 - \frac{7}{3}\right|$

$= \left|\frac{-32}{3}\right|$

$= \frac{32}{3}$ Unit squared

5.5 $\int_{1}^{3}(x^3 - 6x^2 + 11x - 6)dx$

$= \left[\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x\right]_1^3$

$=\left|\left(\frac{2^4}{4} - 2(2)^3 + \frac{11}{2}(2)^2 - 6(2)\right) - (\frac{1^4}{4} - 2(1)^3 + \frac{11}{2}(1)^2 - 6(1))\right| + \left|\left(\frac{3^4}{4} - 2(3)^3 + \frac{11}{2}(3)^2 - 6(3)\right) - \left(\frac{2^4}{4} - 2(2)^3 + \frac{11}{2}(2)^2 - 6(2)\right)\right|$

$= \left|\frac{1}{4}\right| + \left|\frac{-1}{4}\right|$

$= \frac{1}{2}$

5.6 $\int_1^3 (x^2 + 2)\, dx$

$$= \left[\frac{x^3}{3} + 2x\right]_1^3$$

$$= \left|\left(\frac{3^3}{3} + 2(3)\right) - (\frac{1^3}{3} + 2(1))\right|$$

$$= 15 - \frac{7}{3}$$

$$= \frac{38}{3}$$

Revision Exercise

1 $\frac{1}{2} \times \frac{x^2}{2} + C = \frac{x^2}{4} + C$

2 $x^2 + 3x + C$

3 $\frac{4x^3}{3} + \frac{3x^2}{2} - x + C$

4 $x^8 + \frac{3x^2}{2} + C$

5.1 $\int_0^1 (3x^2 + 7x + 1)dx$

$$= \left[\frac{3x^3}{3} + \frac{7x^2}{2} + x\right]_0^1$$

$$= 1 + \frac{7}{2} + 1 = \frac{11}{2}$$

5.2 $\int_0^2 (x - 4x^2)dx$

$$= \left[\frac{x^2}{2} - \frac{4x^3}{3}\right]_0^2$$

$$= (\frac{2^2}{2} - \frac{4(2)^3}{3})$$

$$= 2 - \frac{32}{3} = -\frac{26}{3}$$

5.3 $\int_4^5(-7+6x+x^2)dx$

$$=\left[-7x+\frac{6x^2}{2}+\frac{x^3}{3}\right]_4^5$$

$$=\left[-7(5)+3(5)^2+\frac{5^3}{3}\right]-\left[-7(4)+3(4)^2+\frac{4^3}{3}\right]$$

$$=\frac{245}{3}-\frac{124}{3}=\frac{121}{3}$$

5.4 $\int_{-2}^3(x+2)(x-3)dx$

$$=\left[\frac{x^3}{3}-\frac{x^2}{2}-6x\right]_{-2}^3$$

$$=\left[\frac{3^3}{3}-\frac{3^2}{2}-6(3)\right]-\left[\frac{(-2)^3}{3}-\frac{(-2)^2}{2}-6(-2)\right]$$

$$=\frac{-27}{2}-\frac{22}{3}=-\frac{125}{6}$$

5.5 $\int_{-1}^0(1-x^2)dx$

$$=\left[x-\frac{x^3}{3}\right]_{-1}^0$$

$$=0-(-1-\frac{(-1)^3}{3}$$

$$=0-\left(-1+\frac{1}{3}\right)$$

$$=-\left(\frac{-2}{3}\right)=\frac{2}{3}$$

6.1 $A = -\int_{-2}^{0} x^3 dx + \int_{0}^{2} x^3 dx$

$$= -\left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{2}$$

$$= -\left[0 - \frac{(2)^4}{4}\right] + \left[\frac{(2)^4}{4} - 0\right]$$

$$= -\left[\frac{-16}{4}\right] + \left[\frac{16}{4}\right]$$

$$= \frac{16}{4} + \frac{16}{4}$$

$$= \frac{32}{4}$$

$\therefore = 8$

6.2 $A = \int_{0}^{1} \left(-3x^2 - x^2\right) dx$

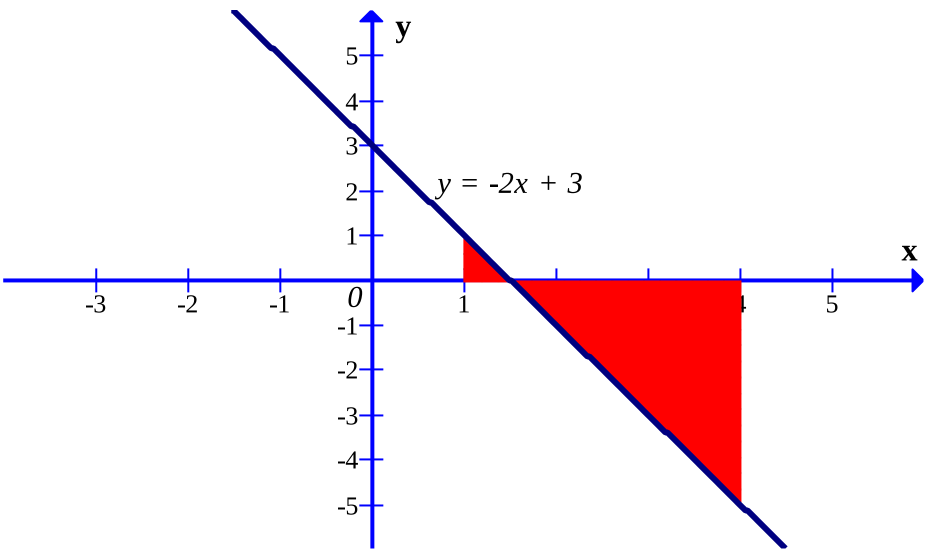
$$= \left[-x^3 - x^2\right]_0^1$$

$$= -\left[-(1)^3 - (1)^2\right] + [0]$$

$$= -[-2]$$

$\therefore = 2$

7.1 $f(x) = -2x + 3$ for $x = 1$ to $x = 4$



$$\int_1^4 (-2x+3)dx$$

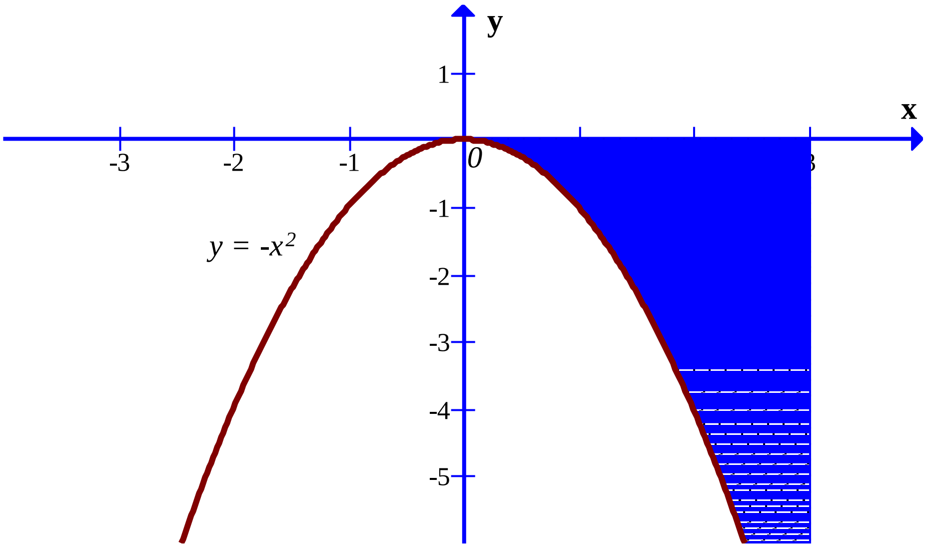
$$= |-x^2 + 3x|_1^4$$

$$= |-x^2 + 3x|_1^{1,5} + |-x^2 + 3x|_{1,5}^4$$

$$= |((-1,5)^2 + 3(1,5)) - (-1+3)| + \left|\left((-4^2) + 3(4)\right) - (1,5)^2 + 3(1,5))\right|$$

$$= \left|\frac{1}{4}\right| + \left|\frac{-25}{4}\right| = \frac{13}{2} \textit{ square units}$$

7.2 $f(x) = -x^2$ for $x = 0$ to $x = 3$

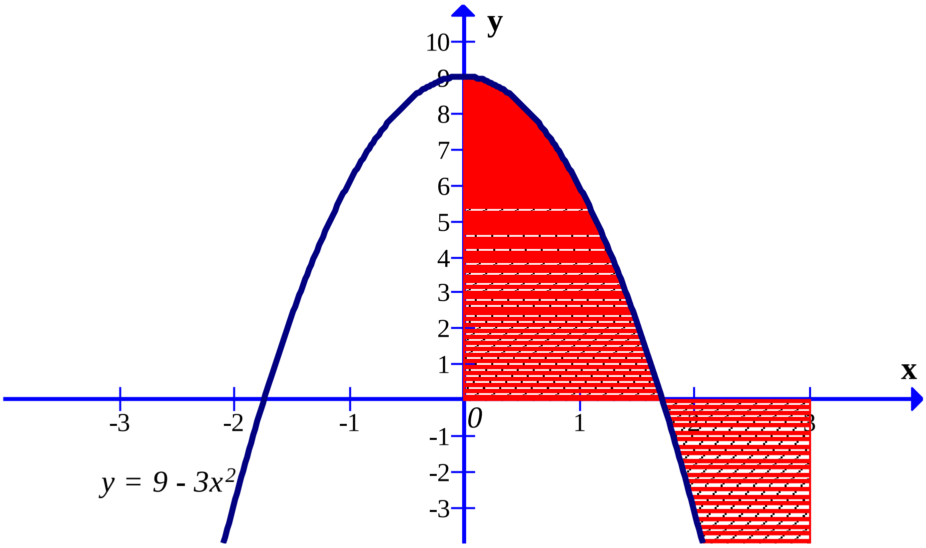


$$\int_0^3 -x^2 dx$$

$$= \left|-\frac{x^3}{3}\right|_0^3$$

$$= \left|\frac{-3^3}{3}\right| = |-9| = 9 \textit{ square units}$$

7.3 $f(x) = 9 - 3x^2$ for $x = 0$ to $x = 3$



$$\int_0^3 (9 - 3x^2)dx$$

$= |9x - x^3|_0^3$

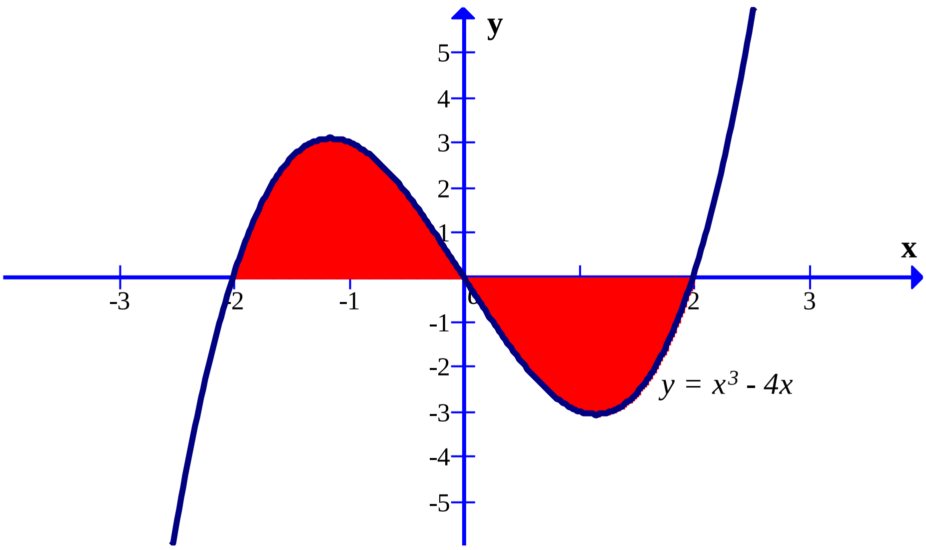
$= |9x - x^3|_0^{1,7} + |9x - x^3|_{1,7}^3$

$= |9(1,7) - 1,7^3| + |(9(3) - 3^3) - (9(1,7) - 1,7^3|$

$= |10,387| + |-10,387|$

$10,387 + 10,387 = 20,774\ \ square\ units$

7.4 $f(x) = x^3 - 4x$ for x = -2 to x = 2



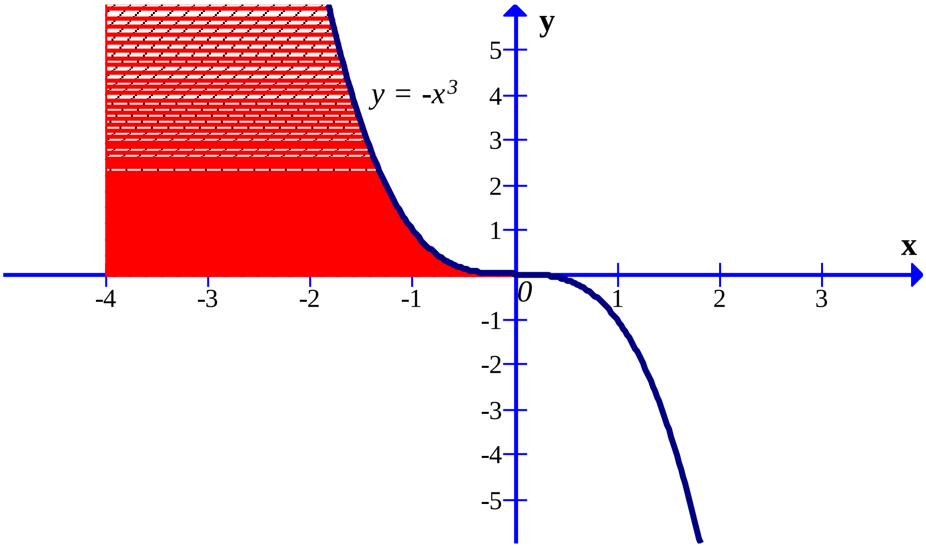
$$\int_{-2}^{2}(x^3-4x)dx = \left|\frac{x^4}{4}-2x^2\right|_{-2}^{2}$$

$$= \left|\frac{x^4}{4}-2x^2\right|_{-2}^{0} + \left|\frac{x^4}{4}-2x^2\right|_{0}^{2}$$

$$= \left|0-(\frac{(-2)^4}{4}-2(-2)^2)\right| + \left|\frac{2^4}{4}2(2)^2\right|$$

$$= |4| + |-4| = 8\ square\ units$$

7.5 $f(x) = -x^3$ for x = -4 to x = 0



$$\int_{-4}^{0}(-x^3)dx$$

$$= \left|\frac{-x^4}{4}\right|_{-4}^{0}$$

$$= \left|0-(\frac{-4^4}{4})\right|$$

= 64 square units