7 TRIGONOMETRY

Objectives

In this chapter you will learn about

- Revision of solution of triangles.
- Solving problems in 2–dimensions and in 3-dimensions by constructing and interpreting models.

7.1 Revision

In previous grades, we dealt with the types of triangles named according to side and angles. In trigonometry, we are able to find the unknown sides and angles when given the magnitudes of the minimum required number of angles and sides.

We looked at two main types of triangles:

- Right angled triangles.
- Non-right angled triangles

Right angled triangles

When a right angled triangle is given, the following may be useful:

- Pythagoras theorem
- Definitions of Trigonometric ratios

Non-right angled triangles

- Sine rule
- Cosine rule
- Area rule

NOTE:

- Not all triangles that you need to solve are right-angled triangles. In grade 11 you learned about rules which can be used to solve non-right-angled triangles.
- Theorems/axioms involving lines, triangles, quadrilateral and circles are also useful.

7.1.1 The Sine Rule

The sine rule expresses the relationship between the two of the sides of the triangles and two of the angles.

In any triangle, *a* is the side opposite angle A, *b* is the side opposite angle B and *c* is the side opposite angle C, then

$$\frac{\text{the lengthof any side}}{\text{the sine of the angle opposite that side}} = \frac{\text{the length of any other side}}{\text{the sine of the angle opposite that side}}$$

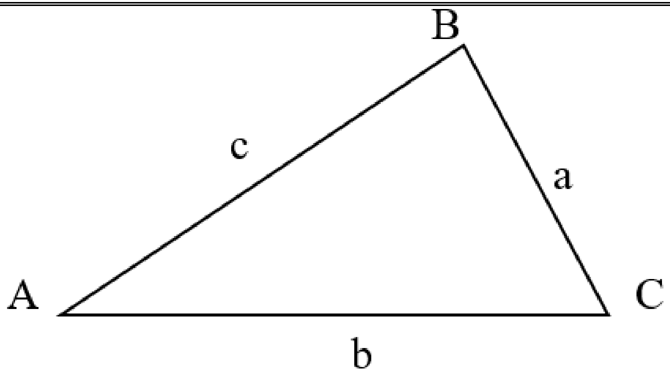
That is:

In Δ ABC,

$$\frac{\mathbf{a}}{\mathbf{\sin A}} = \frac{\mathbf{b}}{\mathbf{\sin B}} = \frac{\mathbf{c}}{\mathbf{\sin C}}$$

This can also be written as:

$$\frac{\mathbf{\sin A}}{\mathbf{a}} = \frac{\mathbf{\sin B}}{\mathbf{b}} = \frac{\mathbf{\sin C}}{\mathbf{c}}$$

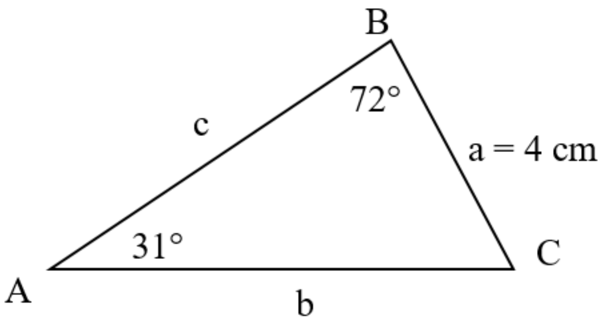


The sine rule can be used when the following information about a triangle is given:

- Two sides and an angle opposite to one of the two sides
- One side and any two angles

Worked Example 1

Solve the triangle given below.



$\hat{C} = 180° - (31° + 72°)$ (sum of angles in a triangle)

$= 180° - 103°$

$= 77°$

To calculate the length of side c use:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a}{\sin A}\sin C$$

$$= \sin 77° \frac{4}{\sin 31°}$$

$= 7,5673$

$c \approx 7,6\ cm$

To calculate the length of side b use:

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$b = \frac{a \sin B}{\sin A}$$

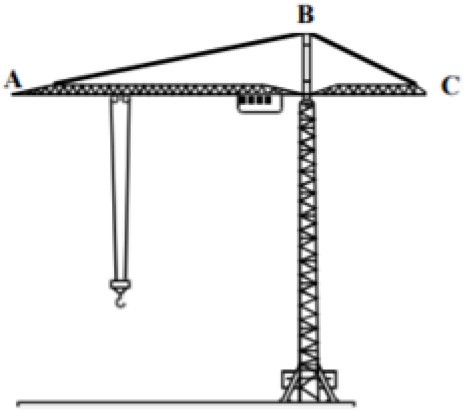
$$b = \frac{4}{\sin 31°} \sin 72°$$

$b = 7,38630064$

$b \approx 7,4\ cm$

Worked example 2

A crane is used in the construction of a shopping complex. AC is the crossbeam and is supported by two metal stays as shown in the sketch below.



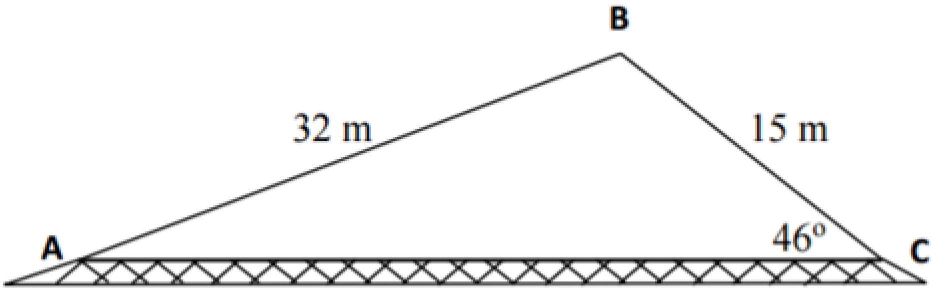
The length of AB is 32 m and the length of BC is 15 m. $B\hat{C}A = 46°$

Calculate:

(a) the size of $B\hat{A}C$

(b) the length of the crossbeam AC

A 2D representation of the given situation



(a) $\dfrac{\sin B\hat{A}C}{15} = \dfrac{\sin 46°}{32}$

$\sin B\hat{A}C = \dfrac{15\sin 46°}{32}$

$= 0,3371905314$

$B\hat{A}C = \sin^{-1}(0,3371905314)$

$= 19,7°$

(b) $A\hat{B}C = 180° - (46° + 19,7°)$

$= 114,3°$

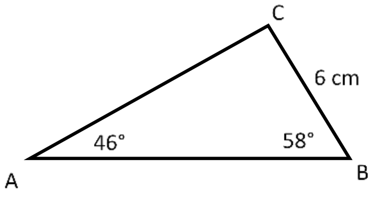
$\dfrac{AC}{\sin 114,3°} = \dfrac{32}{\sin 46°}$

$AC = \dfrac{32\sin 114,3°}{\sin 46°}$

$AC = 40,5\ m$

Exercise 1

1.1 Use the information in ΔABC to find the length of AC. Round off your answer to 1 decimal place.



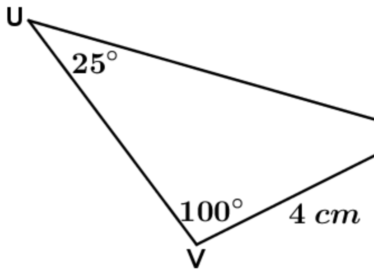
1.2 In ΔPQR angle P = 60°, angle Q = 34° and PQ = 3,8 cm.

Find:

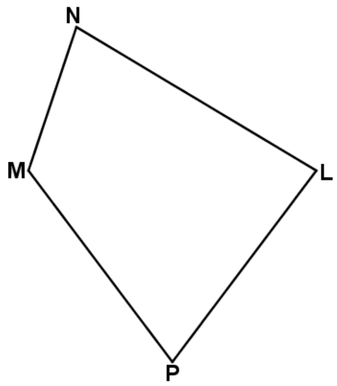
1.1.2 angle R

1.1.3 the length QR.

1.3 Solve Δ UVW



1.4 MNLP represents the playing fields at Marla Secondary School.



$NP = 100\ m$, $P\hat{N}L = 32°$, $N\hat{P}L = 40°$, $N\hat{P}M = 36°$ and $M\hat{N}P = 45°$

Calculate:

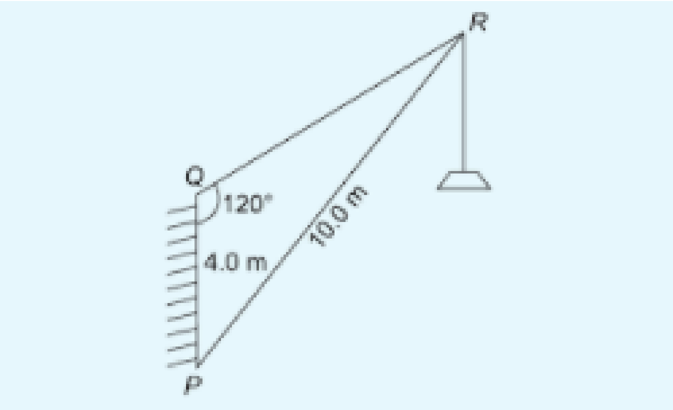
1.4.1 $\hat{L}$

1.4.2 NL

1.4.3 LP

1.4.4 MP

1.5 Consider the figure below. PR is 10 metres long and represents the jib of a crane. The length of PQ is 4 metres.

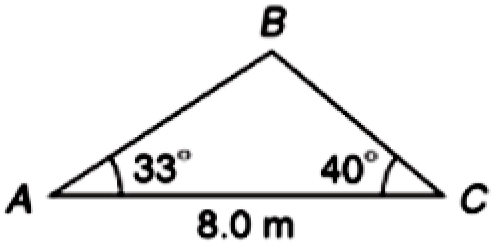


Calculate:

1.5.1 $\hat{P}$ (correct to one decimal place)

1.5.2 the length of QR (correct to two decimal places)

1.6 A room 8,0 m wide has a span roof which slopes which slopes at $33°$ on one side and $40°$ on the other. A section of the roof is shown in the figure below.



Calculate the length of the roof slopes correct to one decimal place.

7.1.2 The Cosine Rule

The cosine rule expresses the relationship between the three sides of a triangle and one of its angles.

The square of one side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides, multiplies by the cosine of the included angle, that is:

In Δ ABC

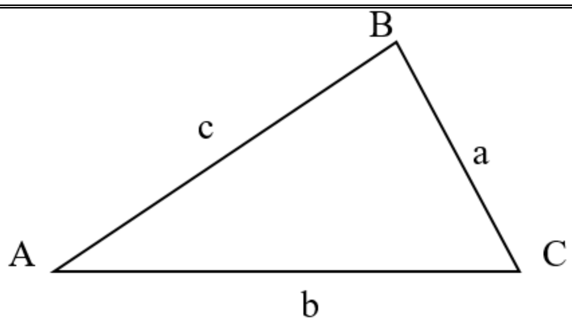
$a^2 = b^2 + c^2 - 2bc\cos A$

Or

$b^2 = a^2 + c^2 - 2ac\cos B$

Or

$c^2 = a^2 + b^2 - 2ab\cos C$



It can also be written in terms of the angle as follows:

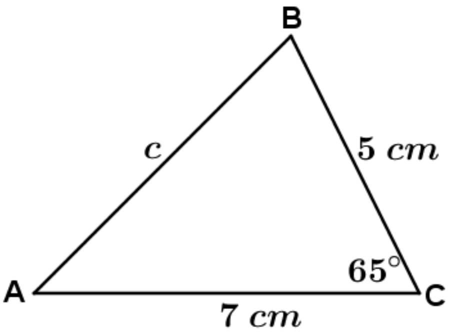
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ or } \cos B = \frac{a^2 + c^2 - b^2}{2ac} \text{ or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine rule can be used when the following information about a triangle is given:

- Two sides and an angle
- Three sides

Worked Example 3

Solve ΔABC.



Two sides and an included angle

$c^2 = a^2 + b^2 - 2ab\cos C$

$c = \sqrt{5^2 + 7^2 - 2(5)(7)\cos 65°}$

$c = 6,66\ cm$

$c \approx 6,7\ cm$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{7^2 + (6,7)^2 - 5^2}{2(7)(6,7)}$$

$= 0,734434968$

$A = \cos^{-1}(0,734434968)$

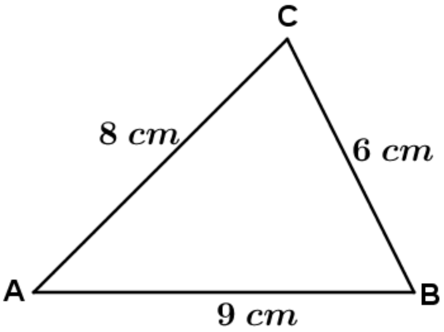
$A = 42,7°$

$B = 180° - (42,7° + 65°)$

$B = 72,3°$

Worked Example 4

Solve $\triangle ABC$.



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{8^2 + 9^2 - 6^2}{2(8)(9)}$$

$$A = \cos^{-1}\left(\frac{8^2 + 9^2 - 6^2}{2(8)(9)}\right)$$

$$A = 40{,}8^\circ$$

Then use the **sine rule to find one of the other angles:**

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 40{,}8^\circ}{6} = \frac{\sin B}{8}$$

$$\sin B = \frac{8 \sin 40{,}8^\circ}{6}$$

$$B = \sin^{-1}\left(\frac{8 \sin 40{,}8^\circ}{6}\right)$$

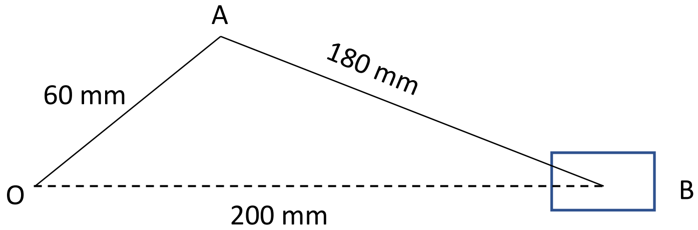
$$B = 60{,}6^\circ$$

$$\hat{C} = 180^\circ - \left(40{,}8^\circ + 60{,}6^\circ\right)$$

$$C = 78{,}6^\circ$$

Worked Example 5

The figure below shows a slider crank mechanism.



The arm is OA and is 60 mm long. The connecting rod, AB, is 180 mm in length. OB is 200 mm long. Calculate the size of $\hat{A}$

$$OB^2 = OA^2 + AB^2 - 2OA.AB\cos A$$

$$200^2 = 60^2 + 180^2 - 2(60)(180)\cos A$$

$$40000 = 3600 + 32400 - 21600\cos A$$

$$40000 = 36000 - 21600\cos A$$

$$21600\cos A = 36000 - 40000$$

$$21600\cos A = -4000$$

$$\cos A = -\frac{4000}{21600}$$

$$\cos A = -0,8151851852$$

The cosine function is negative in the second and third quadrants

$$A = \cos^{-1}(0,8151851852)$$

A in the second quadrant where the cosine is negative

$$A = 180° - \cos^{-1}(0,8151851852)$$

$$A = 144,6°$$

A in the third quadrant where the cosine is negative

$$A = 180° + \cos^{-1}(0,8151851852)$$

$$A = 215,4°$$

This answer is not possible because the sum of angles in a triangle is

$$\therefore A = 144,6°$$

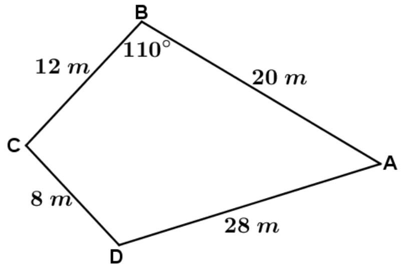
Exercise 2

2.1 In Δ LMN $\hat{L} = 102°$, LM = 10 cm and LN = 12 cm. Find MN.

2.2 In **Δ** PQR, PQ = 1,3 m, PR = 1,4 m and QR = 2,2 m. Find $\hat{R}$.

2.3 A village has 3 main buildings: a church C, a shop S and a bridge B in the shape of a triangle. CB = 120 m, SB = 80 m and $S\hat{B}C$ = 78°. Calculate how far the church is from the shop.

2.4 A piece of land has the figure of a quadrilateral ABCD with $AB = 20\ m$, $BC = 12\ m$, $CD = 7$ m and $AD = 28$ m. The owner decides to divide the land into plots by creating a fence from A to C. It is given that $\hat{B} = 110°$.



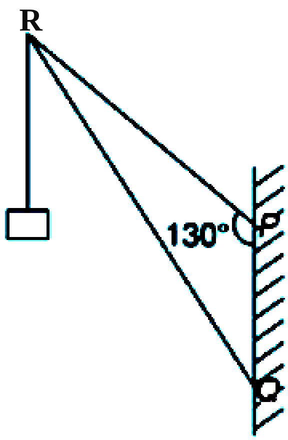
Calculate

2.4.1 the length of the fence AC, correct to 1 decimal place.

2.4.2 The size of $B\hat{A}C$ to the nearest degree.

2.4.3 The size of $\widehat{D}$ to the nearest degree

2.5 A jib crane is shown in the figure below.



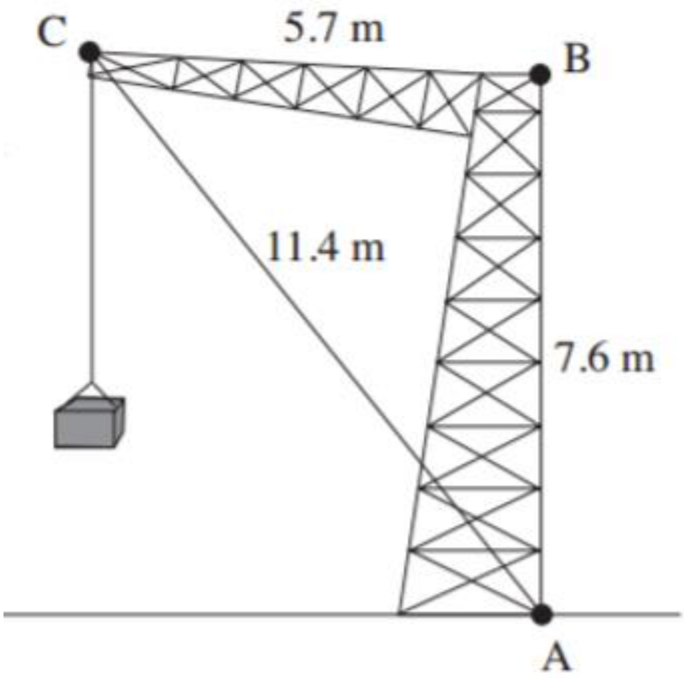
PR is the tie rod and is 8 metres long. The length of PQ is 4 metres.

Calculate

2.5.1 The length of jib RQ

2.5.2 $P\hat{R}Q$

2.6 The diagram below shows a crane offloading cargo from a ship. AB is perpendicular to the ground. CB = 5,7 *m*, AB = 7,6 *m*, and AC =11,4 *m*



Calculate:

2.6.1 $B\hat{A}C$

2.6.2 the height of point C above the ground

7.1.3 The Area Rule

The area rule uses trigonometric ratios to calculate the area of triangle when only the lengths of any two of the sides and the angle ***included*** by these two sides are given.

In Δ ABC,

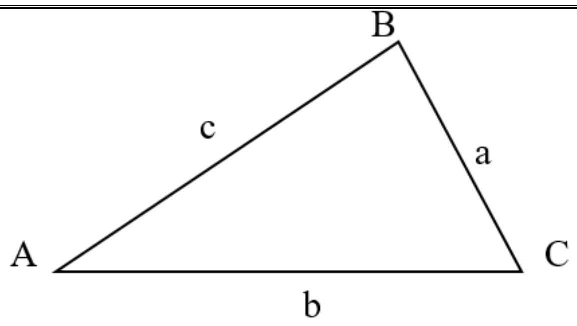
Area $\Delta ABC = \frac{1}{2}bc\sin A$

OR

Area $\Delta ABC = \frac{1}{2}ac\sin B$

OR

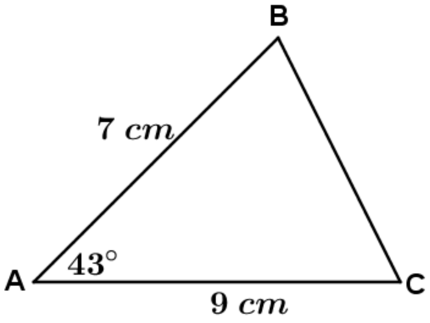
Area $\Delta ABC = \frac{1}{2}ab\sin C$



NOTE : We can use the area rule when we are given two sides of a triangle and the included angle.

Worked Example 6

Calculate the area of ΔABC.



$$\text{Area } \Delta ABC = \frac{1}{2} bc \sin A$$

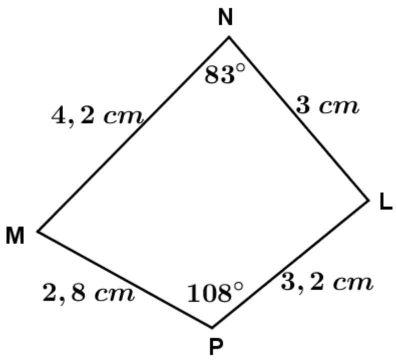
$$= \frac{1}{2}(7)(9)\sin 43°$$

$$= 21{,}5\ cm^2$$

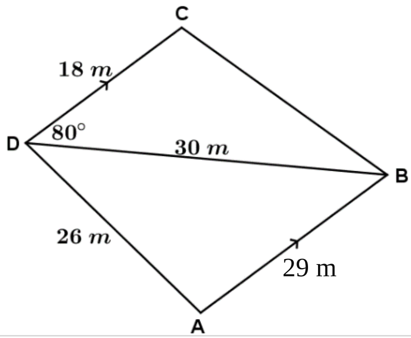
Exercise 3

3.1 Calculate the area of an equilateral triangle of side 10 cm.

3.2 Calculate the area of the quadrilateral MNLP.



3.3 The diagram below shows the plan of a garden (not drawn to scale). The garden is a trapezium with $AD = 26\ m$, $AB = 29\ m$, $DC = 18\ m$ and $D\hat{B}C = 80°$. A straight path from B to D has a length of 30 m.



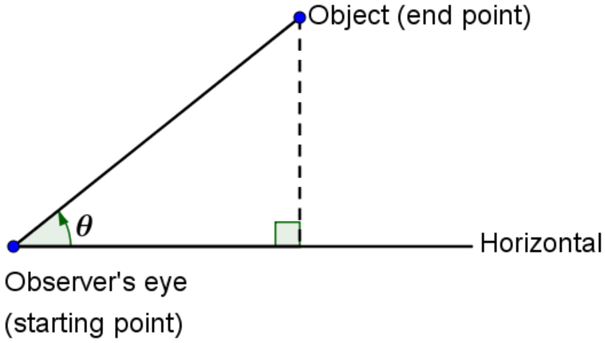
Calculate:

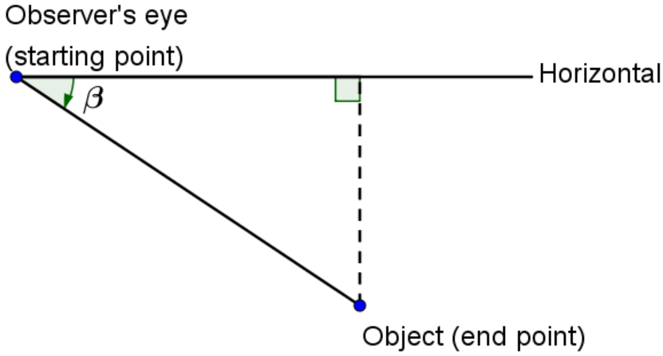
3.3.1 $D\hat{A}B$

3.3.2 The area of the garden to the nearest hundred.

7.1.4 Angles of elevation and depression

To solve problems involving angles of elevation and depression, we need to know the horizontal line (observer's original eye level) and the non-horizontal line (observer's lowered/raised eye level).

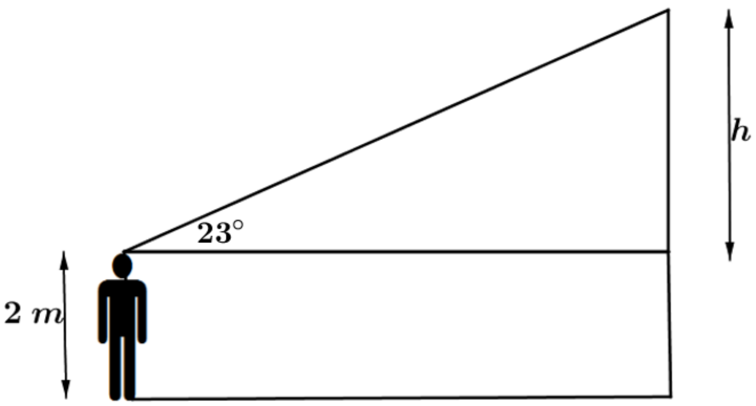




- If the endpoint is above the starting point, then we have an angle of elevation.
- If the endpoint is below the starting point, then we have an angle of depression.

Worked Example 7

A man, 2m tall stands on horizontal ground 30 m from a tree. The angle of elevation of the top of the tree from his eyes is 23°. Estimate the height of the tree correct to one decimal place.



$$\tan 23° = \frac{h-2}{30}$$

$$h - 2 = 30\tan 23°$$

$$h = 2 + 30\tan 23°$$

$h = 14,7\ m$

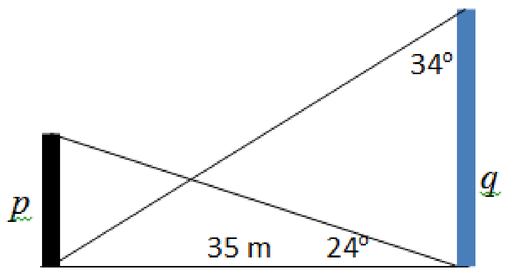
Exercise 4

4.1 An observer standing on top of a vertical cliff spots a house in the adjacent valley at an angle of depression of 12°. If the cliff is 60 m tall, how far is the house from the base of the cliff?

4.2 A 6 m pole AB stands vertically on the ground. Point C is at the angle of depression of 53° from B.

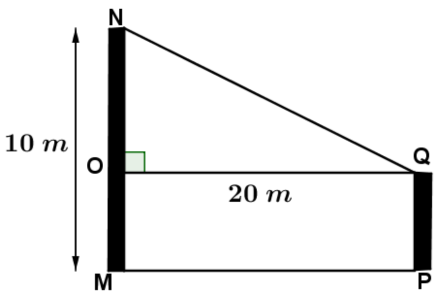
Calculate the distance BC.

4.3 Buildings ***p*** and ***q*** are across the street from each other, 35 *m* apart. From a point on the roof of building ***p***, the angle of elevation at the top of building ***q*** is 24° and the angle of depression of the base of building ***q*** is 34° as shown in the diagram below:



How tall is each building?

4.4 MN and PQ are two poles that stand vertically on flat ground. $MN = 10\ m$ and $PQ = 4\ m$. The distance between these two poles is 20 m. Calculate the angle of elevation of N from Q, correct to 1 decimal place.



7.2 Solving Problems in 2-D and 3-D using Trigonometry

To solve problems in 2 and 3-dimensions you need to:

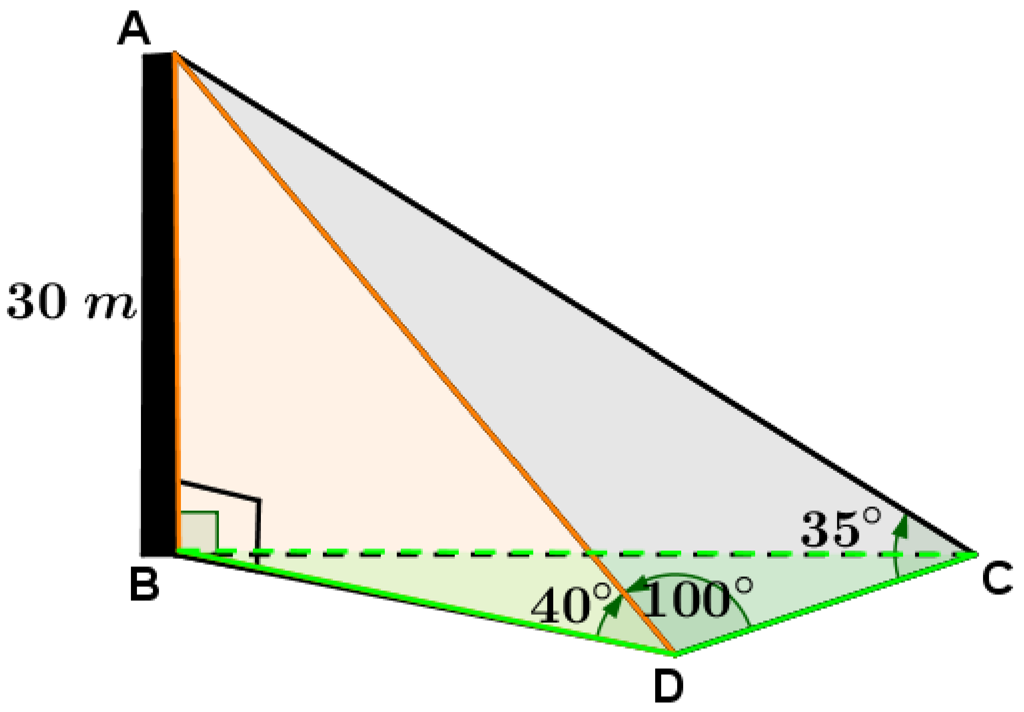
- interpret the given diagrams or sketches.
- identify the triangles you need to work in.
- decide which trigonometric ratios or rules to use to find the necessary information.
- Use Pythagoras theorem if a right-angled triangle is given.

Worked example 8

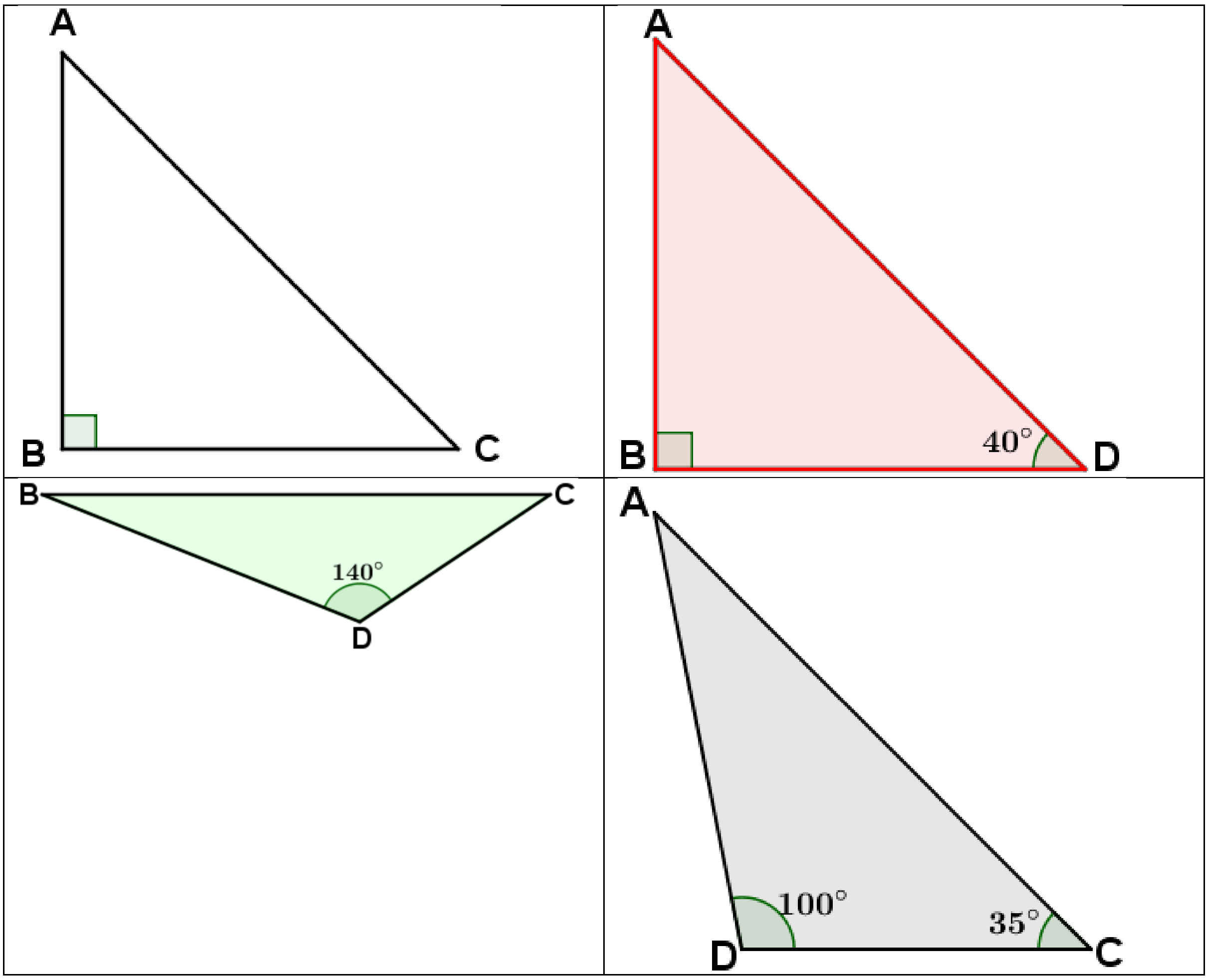
AB is a vertical tower. C and D are points in the same horizontal plane as B, the foot of the tower. AB is 30 m and the angle of elevation of A, measured at D is 40°. $A\hat{C}D = 35°$, $A\hat{D}C = 43°$.

a) Calculate the distance between the points C and D.
b) Determine the area of triangle ADC

Let us sketch the diagram



We can identify the following triangles from the sketch.



a) **We start with ΔABD:**

$$\sin 40° = \frac{30}{AD}$$
$$AD = \frac{30}{\sin 40°}$$
$$\therefore AD = 46,67\ m$$

Now, in ΔADC:

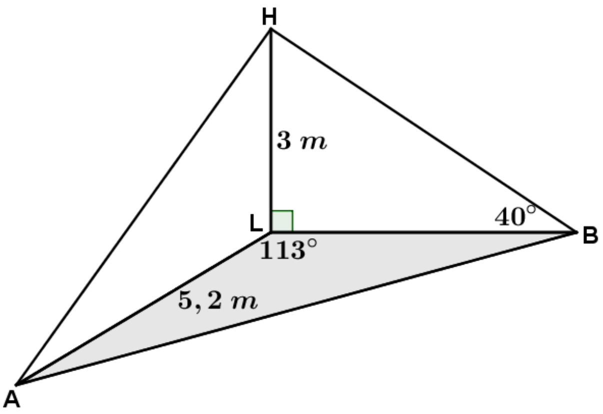
$$\frac{DC}{\sin 100°} = \frac{AD}{\sin 35°}$$
$$DC = \frac{46,67 \sin 100°}{\sin 35°}$$
$$DC = 80,13\ m$$

b) Area $\Delta ADC = \frac{1}{2} AD \times DC \sin 100°$

$$= \frac{1}{2} \times 46,67 \times 80,13 \sin 100°$$
$$= 1841,43\ m^2$$

Exercise 5

5.1 A, B and L are points in the same horizontal plane, HL is a vertical pole of length 3 metres, AL = 5,2 m, $A\hat{L}B = 113^{o}$ and the angle of elevation of H from B is 40°.

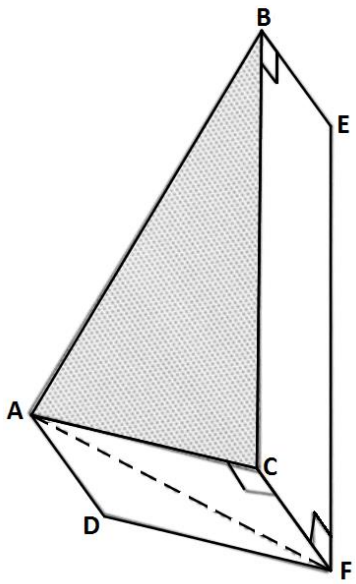


5.1.1 Calculate the length of LB.

5.1.2 Hence, or otherwise calculate the length of AB.

5.1.3 Calculate the area of ΔABL .

5.2 The figure below represents a triangular prism with BA = BC = 5 units, $A\hat{B}C = 50°$ and $F\hat{A}C = 25°$.

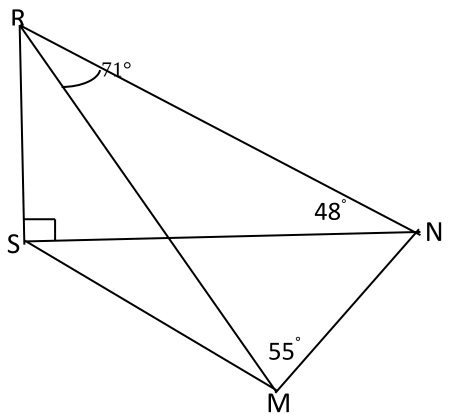


5.2.1 Determine the area of triangle ABC.

5.2.2 Calculate the length of AC, correct to 1 decimal place.

5.2.3 Hence determine the height FC of the prism. Round off your answer to 1 decimal place.

5.3 In the figure below, Pheku is standing at point R on top of building RS that is 50 m high. He observes two cars, M and N, which are in the same horizontal plane as S. The angle of elevation from M to R is 55° and the angle of elevation from N to R is 48°. Angle $M\hat{N}R = 71°$.

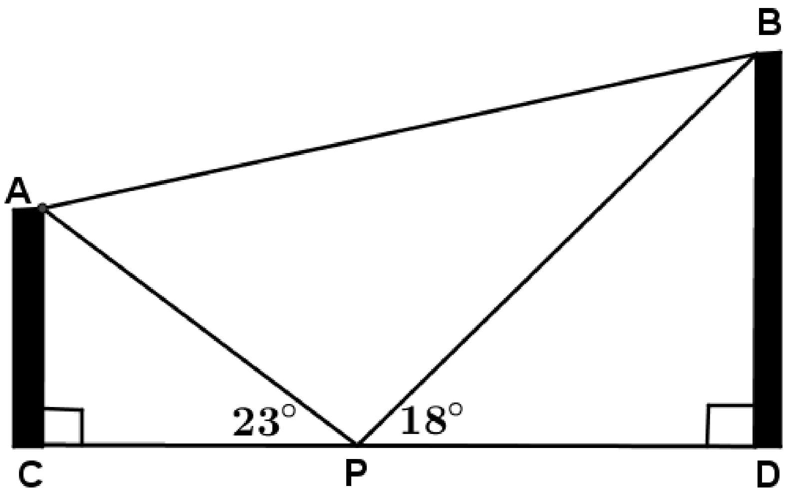


5.3.1 Calculate the lengths of RN, correct to 1 decimal place

5.3.2 Calculate the distance between the two cars that is the length of MN.

5.3.3 Calculate the area of ΔRSN.

5.4 May is standing at a point P on the horizontal ground and observes two poles of different height, AC and BD. P, C and D are in the same horizontal plane. From P, the angle of inclination to the top of the poles A and B are 23° and 18° respectively. May is 18 m from the base of the pole AC. The height of the pole BD is 7 m.



Calculate the following, rounding your answers to 2 decimal places:

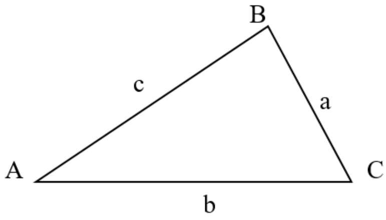
5.4.1 The distance from May to the top of pole BD.

5.4.2 The distance from May to the top of pole AC.

5.4.3 The distance between the top of the poles, that is the length of AB, if $A\hat{P}B = 42°$.

Chapter Summary

- Solving a triangle means finding the missing sides and angles.



- **The Sine rule** states that in any Δ ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

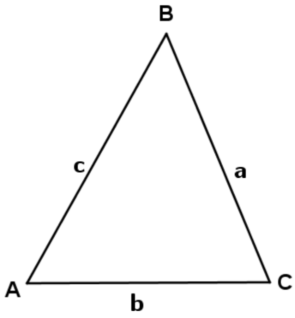
  This can also be written as: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

- **The Cosine rule** states that in any triangle ABC:

  $\therefore c^2 = a^2 + b^2 - 2ab\cos\hat{C}$

  $\therefore b^2 = a^2 + c^2 - 2ac\cos\hat{B}$

  $\therefore a^2 = b^2 + c^2 - 2bc\cos\hat{A}$



Rearranging the above, we have:

$$\cos\hat{B} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\hat{C} = \frac{a^2 + b^2 - c^2}{2ab}$$

- **The area rule** is one way to calculate the area of any triangle.

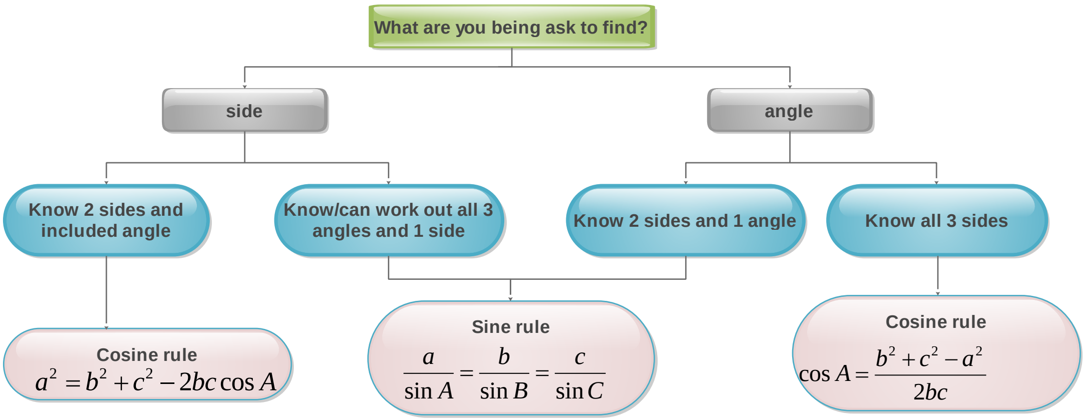
  Area of $\triangle ABC = \frac{1}{2}bc\sin A$

  or

  Area of $\triangle ABC = \frac{1}{2}ac\sin B$

  or

  Area of $\triangle ABC = ab\sin C$



- **Heights and Distances in 3 Dimensions**

When working with questions involving heights and distances in trigonometry, it is important to be able to **visualise** the problems, **to make a rough sketch** of the situation described and to **draw all the different triangles** that are involved in the problem. This will enable you to **DECIDE** which trigonometric formulae to use.

Revision Exercise

1. In triangle XYZ angle X = 132°, angle Z = 21° and YZ = 12 cm. Find the length of XY.

2. In Δ ABC, AB = 5 cm, AC = 6 cm and angle A = 48°. Find the area of Δ ABC.

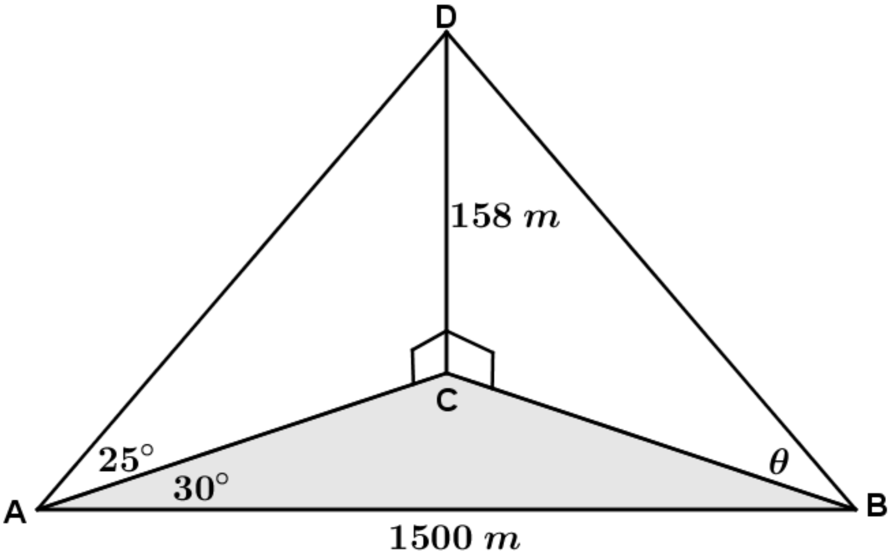
3. A man observes that the angle of elevation of the top of a tower is 25° from a point O, and the distance to the foot of the tower on the level ground is 100m. He walks 40m towards the tower to a point P. the height of the man is 1,5m.

3.1 Calculate the height of the tower, giving your answer to 1 decimal place

3.2 Calculate the angle of elevation of the top of the tower from P.

3.4 Hence, state the angle of depression from the top of the tower to point P.

4. In the diagram below, AB is a straight line 1500 m long, DC is a vertical tower 158 m high with points C, A and B on the same horizontal plane. The angles of elevation of D from A and B are 25° and θ° respectively. It is also given that $C\hat{A}B = 30°$.



4.1 Determine the length of AC

4.2 Determine the length of BC

4.3 Find the value θ.

4.4 Calculate the area of ΔABC

4.5 Find the length of AD.

4.6 Calculate the size of $A\widehat{D}C$.